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On the generalized self-consistent model for elastic wave propagation in composite materials

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Abstract

The generalized self-consistent model (GSCM) for the analysis of elastic wave propagation in composite materials is recast. Following the idea of the GSCM in the static problem, elastic wave energy in the model is evaluated by applying the energy theorem for the elastic wave scattering in an absorbing medium. The conditions for dynamic effective medium are then obtained in a self-consistent way as those under which the extinguished wave energy in the model vanishes, thus without relying on the multiple scattering formalism. It is shown that the present dynamic GSCM is equivalent to the models of Yang and Mal [J. Mech. Phys. Solids 42 (1994) 1945] and Yang [J. Appl. Mech. 70 (2003) 575] that have been obtained through the use of the multiple scattering formalism of Waterman and Truell. Numerical results for both fiber-reinforced and particulate composites are presented. Physical realizability of the dynamic GSCM in the low frequency limit is discussed briefly.

Keywords: Generalized self-consistent model (GSCM); Elastic wave propagation; Micro-mechanics; Composite materials; Multiple scattering

1. Introduction

The analysis of elastic wave propagation in inhomogeneous media for determining their overall dynamic properties has been investigated by many authors for several decades due to its necessity, for example, in the ultrasonic nondestructive evaluation (NDE) of the composite materials. Elastic waves propagating in such media inevitably undergo multiple scattering by distributed inhomogeneities, which results in dispersion and attenuation of the coherent wave. Due to the inherent complexity of the multiple scattering phenomenon, analytical modeling of wave propagation in the inhomogeneous media is intractable. A brief review of different proposed theories can be found in Yang and Mal (1994) and comparisons between them have been made with their numerical results (Kim, 1996, 2003a).

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On the other hand, numerous model-based approaches have been developed in the area of the micromechanics of composite materials for the determination of static effective properties (Nemat-Nasser and Hori, 1999). Among many others, the generalized self-consistent model (GSCM) proposed by Christensen and Lo (1979) and validated later by Christensen (1990) has been received considerable attention in the recognition that it is the most reliable model to predict static effective properties of composites. For this reason, the GSCM has been studied extensively and extended to different types of composites (Jiang et al., 2001; Huang and Hu, 1995), and also to composites with imperfections or damage (Teng, 1992; Benveniste, 1985).

An extension of the GSCM to the dynamic (elastic wave propagation) problem has been attempted by Yang and Mal (1994) for a fiber-reinforced composite and by Kim (1994) and Yang (2003) for a particulate composite. Yang and Mal (1994) and Yang (2003) implemented the multiple scattering theory of Waterman and Truell (1961) in the frame work of the GSCM and obtained the formulae for the dynamic effective medium in a self-consistent form of the multiple scattering formulae which were originally in non-self-consistent forms. Huang and Rokhlin (1995) and Yang and Mal (1996) applied the dynamic GSCM to coated fiber-reinforced composites to investigate the effect of interphase weakening on the wave propagation. Kim (1994) has extended the GSCM by attempting to find effective material properties so that the total scattered power (or energy) in the model vanishes on the average. The fundamental idea was similar to that of the present paper but the extension was made only in the low frequency region. Several different energy-based formulations of the dynamic effective properties of inhomogeneous materials have been proposed for electromagnetic (Stroud and Pan, 1978; Niklasson et al., 1981; Niklasson and Granqvist, 1984) and elastic (Kim et al., 1995; Kim, 2003a) wave propagation problems. In spite of different assumptions and theoretical backgrounds, the underlying ideas of these theories are all closely related in that the effective medium is defined in terms of the forward scattering amplitude for the constituents embedded in the yet-unknown effective medium. Most recently, Kanaun and Levin (2003) reviewed and further developed three effective medium theories for axial shear wave propagation in fiber-reinforced composites. The present dynamic model corresponds to the third kind effective medium theory in Kanaun and Levin (2003).

In this paper, the generalized self-consistent model for the analysis of elastic wave propagation in the composite materials is recast. The self-consistency of the dynamic effective medium is that the energy in the model is the same with the energy in the effective homogeneous medium when they are under the same dynamic loading. The extinguished energy in the model is then evaluated using the recently derived energy theorem for the associated scattering problem, based on which the self-consistency conditions for effective properties are drawn. A comparison to the results from Yang and Mal (1994) and Yang (2003) is presented. Numerical results are given for different cases of fiber-reinforced and particulate composites. GSCM with an alternative structure in which the materials for the core and the outer shell are switched is examined for its dynamic behavior. Physical realizability of the dynamic model is discussed briefly.

2. Dynamic GSCM

Consider a three-phase geometrical model for the fiber-reinforced and particulate composite materials as shown in Fig. 1. The cylindrical (or spherical) inclusion of radius a is embedded in a concentric annulus (or shell) of the matrix material of radius b, which in turn is embedded in an infinitely extended effective medium that has yet-unknown material properties. The radius b of the composite inclusion is set for the prescribed inclusion volume fraction to be $v_f = a^2/b^2$ for a fiber-reinforced composite and $v_f = a^3/b^3$ for a particulate composite. Although Fig. 1 represents for the composites with cylindrical and spherical inclusions, the theory presented below is not limited to only these geometries but applicable, in principle, to any other inclusion shapes.

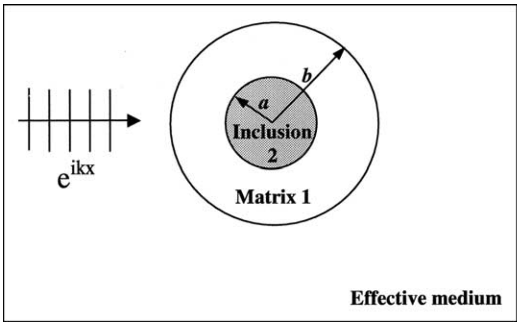

Fig. 1. The generalized self-consistent model for the dynamic problem.

Suppose now a plane elastic longitudinal or shear wave propagates in the effective medium surrounding the embedded composite inclusion as shown in Fig. 1. The incident wave is assumed to be equal to the mean wave field in the composite material by a loading applied at infinitely large distance. To derive the expressions for the effective dynamic properties, the total energy ($\mathbf{U}$) of the elastic wave field in the model is calculated. As in the static GSCM, the total energy is expressed as the sum of the energy ($\mathbf{U}_0$) in the same medium without the composite inclusion by the external loading and the disturbed energy ($\tilde{\mathbf{U}}$) due to the presence of the embedded composite inclusion,

$$\mathbf{U} = \mathbf{U}_0 - \tilde{\mathbf{U}}. \tag{1}$$

It should be noted that $\mathbf{U}_0$ and $\tilde{\mathbf{U}}$ correspond simply to the energy of the incident wave field and the extinguished energy ($\mathbf{U}^{\text{ext}}$) of the scattering problem. Now, a self-consistency of the effective medium is stated that the total energy in the model is the same with the energy of the mean wave field in the composite material, namely, the energy of the incident wave in the homogeneous effective medium under the same dynamic loading, which can be written

$$\mathbf{U} = \mathbf{U}_0, \tag{2}$$

and equivalently from Eq. (2)

$$\tilde{\mathbf{U}} = \mathbf{U}^{\text{ext}} = 0. \tag{3}$$

Note that the present procedure is exactly analogous to that of the GSCM for static effective properties (Christensen and Lo, 1979; Christensen, 1990) and thus does not rely on the multiple scattering formalism of Waterman and Truell (1961) as in Yang and Mal (1994) and Yang (2003).

Eq. (3) can be written equivalently in terms of the extinction cross-sections (the extinguished power normalized by the intensity of the incident wave) of the composite inclusion in the model:

$$\Sigma_{\text{L}}^{\text{ext}} = 0, \tag{4}$$

$$\Sigma_{\text{SV}}^{\text{ext}} = 0, \tag{5}$$

$$\Sigma_{\text{SH}}^{\text{ext}} = 0, \tag{6}$$

where $\Sigma_{\text{L}}^{\text{ext}}$, $\Sigma_{\text{SV}}^{\text{ext}}$, and $\Sigma_{\text{SH}}^{\text{ext}}$ are the extinction cross-sections for the longitudinal and vertically polarized shear (SV) and horizontally polarized shear (SH) waves, respectively. It is noted that the above formulae are the same with those of Niklasson et al. (1981) and Niklasson and Granqvist (1984) for the electromagnetic wave propagation, which were proposed, however, only on the purely physical ground.

Since the effective medium is presumed to be mechanically equivalent to the actual composite medium, the medium in the associated scattering problem has to be energy-absorbing (even when there is no absorption in the constituents) for taking the attenuation due to incoherent scattering into account. The associated scattering problem is, therefore, defined to be a scattering by the composite inclusion in an absorbing surrounding medium. For this reason, the extinguished power (the extinction cross-section), which accounts for both the effects of the scattering and the absorption, is the relevant physical quantity to calculate the total power disturbance by the composite inclusion rather than the scattered power (the scattering cross-section). Kim (2003b,c) has recently obtained the expressions of the generalized extinction cross-sections for the elastic wave scattering in an absorbing medium,

$$\Sigma_i^{\text{ext}} = -4Re\left[\frac{f_i(0)}{\langle k_i\rangle}\right], \quad i = \text{L, SV and SH} \tag{7}$$

for a two-dimensional object and

$$\Sigma_i^{\text{ext}} = -4\pi Re\left[\frac{f_i(0)}{\langle k_i\rangle^2}\right], \quad i = \text{L, SV and SH} \tag{8}$$

for a three-dimensional object, where $\langle k_{\text{L}}\rangle$ and $\langle k_{\text{S}}\rangle$ are the wavenumbers of longitudinal and shear waves in the effective medium and $f_{\text{L}}(0)$, $f_{\text{SV}}(0)$ and $f_{\text{SH}}(0)$ are the forward scattering amplitudes of the longitudinal, SV and SH waves, respectively. It is noted that the extinction cross-sections in an absorbing medium are formally the same with those in the lossless medium. For a spherical inclusion, it is noted also that $\Sigma_{\text{SH}}^{\text{ext}} = \Sigma_{\text{SV}}^{\text{ext}}$. Since the wavenumbers ($\langle k_{\text{L}}\rangle$ and $\langle k_{\text{S}}\rangle$) are complex quantities, Eqs. (4)–(6) are, referring to Eqs. (7) and (8), commonly in both two- and three- dimensional spaces, equivalent to

$$f_{\text{L}}(0) = 0, \tag{9}$$

$$f_{\text{SV}}(0) = 0, \tag{10}$$

$$f_{\text{SH}}(0) = 0. \tag{11}$$

These are the formulae for the dynamic effective media of the fiber-reinforced and the particulate composites derived in the GSCM for the elasodynamic problem, while this type of formulation has been used in electromagnetic problems (Niklasson et al., 1981; Niklasson and Granqvist, 1984). When the effective medium is isotropic, Eqs. (10) and (11) are redundant to each other.

Eqs. (9)–(11) appear to be different from those of Yang and Mal (1994) for a fiber-reinforced composite,

$$1 = \left[1 - \frac{2in_0 f_i(0)}{\langle k_i\rangle^2}\right]^2 - \left[\frac{2in_0 f_i(\pi)}{\langle k_i\rangle^2}\right]^2, \quad i = \text{L, SV and SH} \tag{12}$$

and of Yang (2003) for a particulate composite

$$1 = \left[1 + \frac{2\pi n_0 f_i(0)}{\langle k_i\rangle^2}\right]^2 - \left[\frac{2\pi n_0 f_i(\pi)}{\langle k_i\rangle^2}\right]^2, \quad i = \text{L, and S} \tag{13}$$

both of which were obtained based on the Waterman and Truell (1961), and where $f_i(\pi)$ is the backward scattering amplitude of the corresponding wave and n_0 is the number of inclusions in unit volume (area). Because of the simultaneous application of the single scattering approximation in the Waterman–Truell theory and the approximation of the self-consistent embedding scheme in the GSCM, the physical implication of Eqs. (12) and (13) is not quite obvious. On the contrary, the physical implication of Eqs. (11)–(13) is straightforward and easy to understand: the forward scattering amplitude which is proportional to the total extinguished energy due to the embedded composite object, vanishes when there is no scattering thus only when the surrounding medium is the effective medium.

It can be shown on a purely physical basis that the formulae of Yang and Mal (1994) and Yang (2003), namely, Eqs. (12) and (13) reduce to the derived formulae Eqs. (9)–(11). One of sufficient conditions of, for example, the longitudinal wave propagation in a composite with arbitrary shaped inclusions is

$$f_L(0) = 0, \tag{14}$$

and

$$f_L(\pi) = 0. \tag{15}$$

Since the forward scattering amplitude is proportional to the total energy abstracted from the incident wave due to the presence of the object during the course of scattering, the absence of the forward scattering amplitude indicates simply that there is no scattering, and accordingly the backward scattering amplitude should also be absent. Indeed, it is impossible to conceive a real physical situation in which the forward scattering amplitude of a single object disappears while the backward scattering amplitude still exists. Therefore, the latter condition Eq. (15) is redundant to the former Eq. (14), or $f_L(\pi) = \gamma f_L(0)$ for a frequency dependent coefficient $\gamma = \gamma(\omega)$. Substituting this into Eq. (12) or Eq. (13) yields that the only physically meaningful necessary and sufficient condition for the longitudinal wave excitation is $f_L(0) = 0$. This inference works in the same way for the SV and SH wave propagations. The equivalence is thus hypothesized.

The equivalence is shown also numerically. First, the SH wave propagation in SiC-Ti fiber-reinforced composite is considered. The material properties are taken from Yang and Mal (1994) as presented in Table 1. The fiber volume fractions are 15%, 25%, and 35%. The effective dynamic density is assumed to be the volume fraction weighted average: $\langle\rho\rangle = (1 - v_f)\rho_1 + v_f\rho_2$ as in Yang and Mal (1994) and Yang (2003). Note that this assumption may not be correct when densities of the constituents differ larger than an order of magnitude where the inertial effect due to the density mismatch cannot be ignored any more. Fig. 2 shows the effective wave speed normalized by the shear wave speed in matrix and the coherent wave attenuation $4\pi Im[\langle k_{SH}\rangle]/Re[\langle k_{SH}\rangle]$. It is noted that the results from the present theory and from the theory of Yang and Mal (1994) are indistinguishable illustrating their equivalence. Excessively small attenuations are observed in the low frequency region ($k_{S1}a < 1.0$) regardless of the volume fraction. Second, L and S wave propagations in SiC-Al particulate composite with particle volume fraction of 30% are considered. This is the example in Yang's (2003) paper from which the material properties are obtained as shown in Table 1. Fig. 3 shows L and S wave speeds normalized by the corresponding wave speeds in the matrix and coherent attenuations. Results calculated by the present theory and by the theory of Yang (2003) coincide. Finally, the SV wave propagation in the titanium aluminide matrix reinforced by SiC fibers with a carbon-coating layer is considered. The fiber volume fraction is 35% and the thickness of the carbon layer is 5% of the fiber radius (Yang and Mal, 1997). In the case also, the present theory and Yang and Mal (1997) coincide exactly with each other.

Table 1
Material properties used in the calculations

Material	E (GPa)	μ (GPa)	ρ (kg/m^3)
Al	71.1	26.5	2706
Ti (Fig. 2)	120.8	45.9	4540
Ti (Fig. 4)	96.5	37.1	4500
SiC (Figs. 2 and 3)	440.0	188.0	3180
SiC (Fig. 4)	431.0	172.0	3200
Carbon layer	34.5	14.3	1400
Steel	113.2	80.9	7800

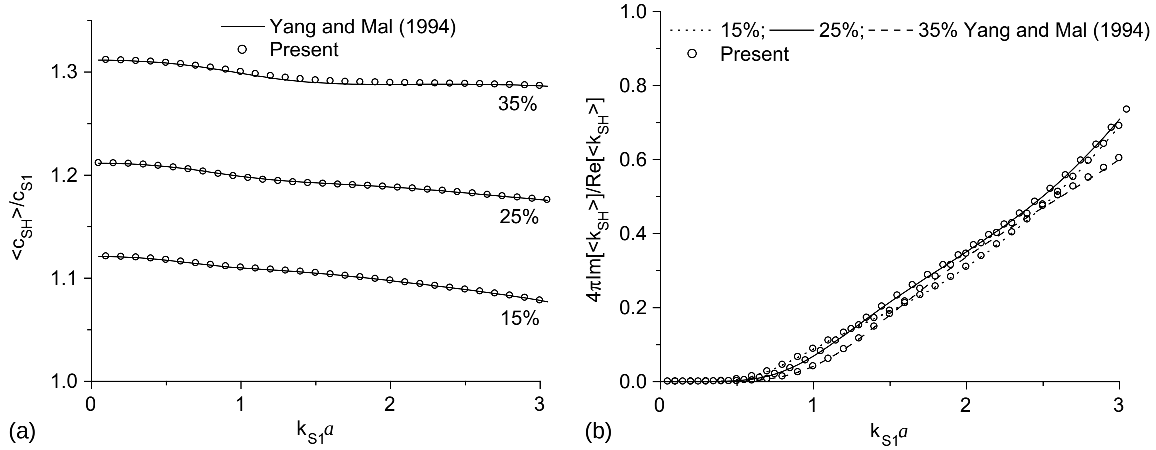

Fig. 2. (a) Normalized effective SH wavespeed in SiC-Ti composite at different fiber volume fractions, calculated by the present theory and by the theory of Yang and Mal (1994). (b) Specific SH wave attenuation capacity of SiC-Ti composite at different fiber volume fractions, calculated by the present theory and by the theory of Yang and Mal (1994).

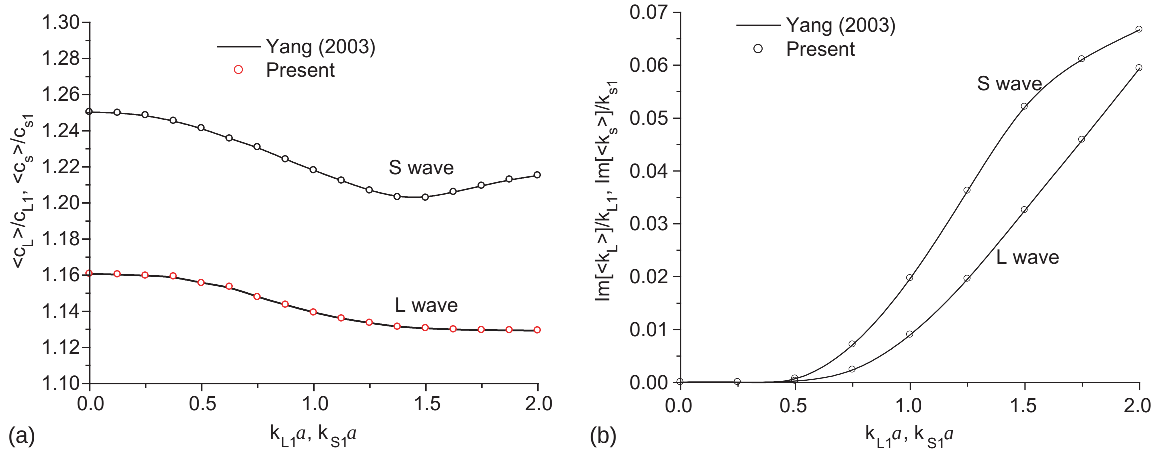

Fig. 3. (a) Normalized effective L and S wavespeeds in SiC-Al particulate composite at Al particle volume fraction 30%, calculated by the present theory and by the theory of Yang (2003). (b) Coherent L and S wave attenuations of SiC-Al particulate composite at Al particle volume fraction 30%, calculated by the present theory and by the theory of Yang (2003).

Another structure of the model alternative to the ordinary one shown in Fig. 1 can be constructed by switching the materials for the core and the concentric annulus (or shell) in the model as shown in Fig. 5. The inner and outer radii a and b are determined by $v_f = (b^2 - a^2)/b^2$ for the two dimensional, and $v_f = (b^3 a^3)/b^3$ for the three dimensional configurations. In the static problem, this alternative model is known to yield effective stiffness values coincident with the rigorous upper bounds (Hashin, 1984) whereas the ordinary model to yield the rigorous lower bounds. Here, the model is examined for its dynamic behavior from numerical results to see if the bounding of the two models consistently exists in the finite

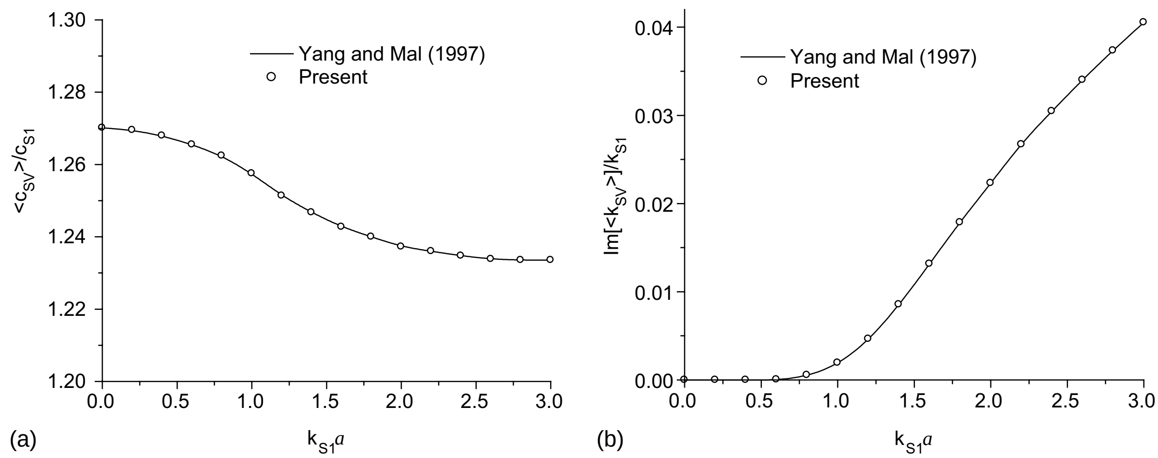

Fig. 4. (a) Normalized effective SV wavespeed in Ti matrix composite reinforced by SiC with carbon coating at fiber volume fraction 35%, calculated by the present theory and by the theory of Yang and Mal (1997). The coating layer thickness (h) to fiber outer radius (a) is $h/a = 0.05$. (b) Coherent SV wave attenuation in Ti matrix composite reinforced by SiC with carbon coating at fiber volume fraction 35%, calculated by the present theory and by the theory of Yang and Mal (1997). The coating layer thickness (h) to fiber outer radius (a) is $h/a = 0.05$.

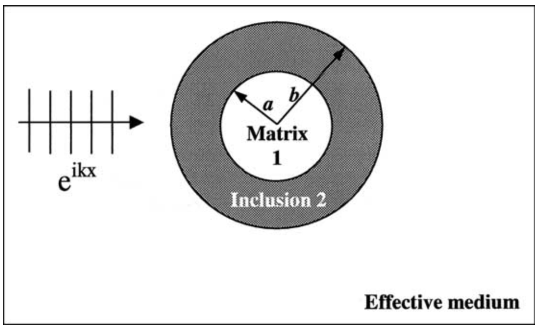

Fig. 5. An alternative structure of the generalized self-consistent model. The materials for the core and the annulus are switched. The radii a and b are determined, $v_f = (b^2 - a^2)/b^2$ for the two dimensional, and $v_f = (b^3 - a^3)/b^3$ for the three dimensional problem.

frequency region. Fig. 6 shows normalized SH wave speeds in steel-Al composite with fiber volume fraction 40% calculated by using both the ordinary and alternative models and the dynamic self-consistent model (SCM) of Kim (2003). Material properties of the steel fiber are shown in Table 1. Consistently to the static case, these dynamic GSCMs nicely bound the SCM in the quasi-static regime ($ka \ll 1$). However, the bounding behavior is not extended to the higher frequency region. With the increase of frequency the alternative GSCM shows much different behavior, furthermore crossing the ordinary GSCM while the ordinary GSCM seems to follow the SCM as a lower bound. The abrupt increase of the wave speed at around $ka = 1$ is know to be due to the simple-oscillator type resonance of heavy inclusions restrained in the matrix (Moon and Mow, 1970; Kim et al., 1995). This motion cannot be described by the structure in the alternative GSCM because the heavy and stiffer inclusion is modeled to be the outer shell, which prevents the formation of the simple oscillator resonance of the inclusions. Therefore, the alternative GSCM, although it yields the upper bound of the effective stiffness in the static limit, does not correctly describe the wave motion in the composite.

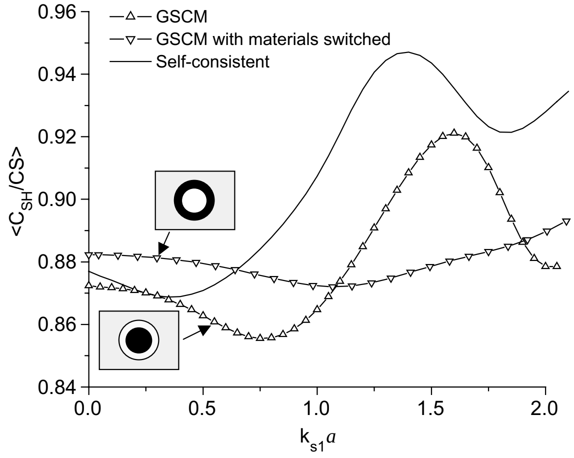

Fig. 6. Normalized effective SH wavespeed in Al matrix composite reinforced by steel fibers of 40% volume fraction. GSCM with two different microstructures are compared with the self-consistent model (Kim, 1996, 2003a).

3. Static limit

It can be shown that in the low frequency limit the derived formulae recover the static effective properties predicted by the static GSCM. The proof of this may be trivial as a matter of complicated algebra. Instead of giving the proof here, an interesting aspect of the dynamic version of the GSCM in the low frequency limit is noted.

Consider the longitudinal wave propagation in a composite material with spherical inclusions (Yang, 2003). The forward scattering amplitude of a spherical composite inclusion for the longitudinal wave scattering in the low frequency limit is given (see Appendix A)

$$f_{\mathrm{L}}(0) \approx \frac{1}{i\langle k_{\mathrm{L}}\rangle}(A_0 + 3A_1 + 5A_2), \tag{16}$$

where i is the unit imaginary number and $A_n\,(n = 0, 1, 2)$ are the lowest order scattering coefficients given as

$$A_0 = \frac{x^3}{3i}\left[\frac{(K_1 - \langle K\rangle) + v_{\mathrm{f}}(K_2 - K_1)\frac{(3\langle K\rangle + 4\mu_1)}{(3K_2 + 4\mu_1)} + O(x^2)}{(K_1 + \frac{4}{3}\langle\mu\rangle) + 4v_{\mathrm{f}}(K_2 - K_1)\frac{(\mu_1 - \langle\mu\rangle)}{(3K_2 + 4\mu_1)} + O(x^2)}\right], \tag{17}$$

$$A_1 = \frac{x^3}{9i}\left[1 - \frac{\rho_1}{\langle\rho\rangle} - v_{\mathrm{f}}\frac{\rho_2 - \rho_1}{\langle\rho\rangle} + O(x^2)\right], \tag{18}$$

$$A_2 = \frac{4ix^3}{3}\langle\mu\rangle\left[\frac{(\mu_1 - \langle\mu\rangle) - v_{\mathrm{f}}(\mu_1 - \mu_2)\frac{\mu_1(9K_1 + 8\mu_1) + 6\langle\mu\rangle(K_1 + 2\mu_1)}{\mu_1(9K_1 + 8\mu_1) + 6\mu_2(K_1 + 2\mu_1)} + O(x^2)}{\langle\mu\rangle(9\langle K\rangle + 8\langle\mu\rangle) + 6\mu_1(\langle K\rangle + 2\langle\mu\rangle) - v_{\mathrm{f}}(\mu_1 - \mu_2)P + O(x^2)}\right], \tag{19}$$

$$P = 6\frac{\langle\mu\rangle(K_1 + 2\mu_1)(9\langle K\rangle + 8\langle\mu\rangle) - \mu_1(\langle K\rangle + 2\langle\mu\rangle)(9K_1 + 8\mu_1)}{\mu_1(9K_1 + 8\mu_1) + 6\mu_2(K_1 + 2\mu_1)},$$

$$A_n = O(x^4), \quad n \geqslant 3$$

where $K_{1,2}$ and $\mu_{1,2}$ the bulk and shear moduli of the constituents 1 and 2, quantities in $\langle\ \rangle$ are those of effective medium, and the parameter $x = \langle k_L \rangle b$. To the best of author's knowledge, the above asymptotic scattering coefficients have not been presented in the literature so far.

Eq. (14) is satisfied only when $A_0 = A_1 = A_2 = 0$ respectively, since these are coefficients of orthogonal spherical harmonics (Ying and Truell, 1956). Therefore, neglecting the higher-order terms in the above coefficients, the effective bulk modulus, density and shear modulus are obtained explicitly

$$\langle K \rangle = K_1 + \frac{v_f(K_2 - K_1)}{1 + (1 - v_f)(K_2 - K_1)/(K_1 + 4/3\mu_1)}, \tag{20}$$

$$\langle \rho \rangle = v_f \rho_2 + (1 - v_f)\rho_1, \tag{21}$$

$$\langle \mu \rangle = \mu_1 \frac{\left[1 + \frac{6\mu_2(K_1+2\mu_1)}{\mu_1(9K_1+8\mu_1)} - v_f\left(1 - \frac{\mu_2}{\mu_1}\right)\right]}{\left[1 + \frac{6\mu_2(K_1+2\mu_1)}{\mu_1(9K_1+8\mu_1)} + v_f \frac{6(K_1+2\mu_1)}{(9K_1+8\mu_1)}\left(1 - \frac{\mu_2}{\mu_1}\right)\right]}, \tag{22}$$

where the constituent 1 denotes the matrix of the composites. First of all, it is noted that these effective moduli are identical to those of Kuster and Toksoz (1974) which are the rigorous lower bounds of Hashin and Shtrikman (1963). While the effective bulk modulus coincides with that from the static GSCM (Christensen and Lo, 1979; Christensen, 1990), the effective shear modulus (a cross modulus) does not. As a matter of course, the low frequency limit result for the shear wave incidence case gives the effective shear modulus (Yang and Mal, 1994) of the static GSCM (Christensen and Lo, 1979; Christensen, 1990). The model thus provides two effective shear moduli dependently on the type of the excitation wave for a single effective medium. This fact leads to the nonuniqueness of the effective medium when it is defined through the GSCM in the finite frequency region. Therefore, the effective medium in the dynamic GSCM is questioned of its physical realizability. The physical realizability of micromechanical models has been discussed by Berryman and Berge (1996). It was noted that the existence of scattering analog is a necessary condition for the realizability but not sufficient to guarantee it. This issue requires further investigation.

4. Summary

The generalized self-consistent model for the elastic wave propagation in composite materials is recast. The self-consistent conditions for the effective medium are derived, resulting in the forms different from those of the existing theory, from the energy consideration analogously to the static GSCM. The derived formulae are shown numerically to be equivalent to those in the theory of Yang and Mal (1994) and Yang (2003) but the present ones have self-obvious physical meaning as well as are straightforward compared to the existing ones. The present model is general in that the frequency dependent electromagnetic properties of the same composite can be obtained equally by finding the properties that make the forward scattering amplitude of electromagnetic waves by the same composite inclusion to be zero. GSCM with an alternative structure seems to be unable to describe correctly the dynamic behavior of inclusions in a composite medium. In the low frequency limit, the dynamic GSCM produces two effective shear moduli, which raises the physical realization problem of the model.

Appendix A. Low frequency spherical GSCM for longitudinal elastic wave propagation

The problem of the longitudinal wave scattering by a concentric composite spherical inclusion shown in Fig. 1 is analyzed briefly. The longitudinal and shear waves are represented by their displacement potentials Φ and Ψ (Ying and Truell, 1956) that satisfy the scalar wave equations

$$(\nabla^2 + k_{\mathrm{L}}^2)\Phi = 0, \tag{A.1}$$

$$(\nabla^2 + k_{\mathrm{S}}^2)\Psi = 0, \tag{A.2}$$

where $k_{\mathrm{L}} = \omega/\{(\lambda + 2\mu)/\rho\}^{1/2}$ and $k_{\mathrm{S}} = \omega/(\mu/\rho)^{1/2}$ are the wavenumbers of longitudinal and shear waves, λ, μ are Lame elastic constants, ω is the angular frequency and ρ is the material density. The incident plane longitudinal wave can be expressed as

$$\Phi_i = \sum_{n=0}^{\infty} i^n(2n+1)j_n(\langle k_{\mathrm{L}}\rangle r)P_n(\cos\theta), \tag{A.3}$$

where j_n is the spherical Bessel function of order n and P_n is the Legendre polynomials. The potentials for scattered waves in the effective medium are given

$$\Phi_e = \sum_{n=0}^{\infty} i^n(2n+1)A_n h_n(\langle k_{\mathrm{L}}\rangle r)P_n(\cos\theta), \tag{A.4}$$

$$\Psi_e = \sum_{n=0}^{\infty} i^n(2n+1)B_n h_n(\langle k_{\mathrm{S}}\rangle r)P_n(\cos\theta), \tag{A.5}$$

where h_n is the first kind spherical Hankel function of order n. The waves in the shell are represented by

$$\Phi_1 = \sum_{n=0}^{\infty} i^n(2n+1)[C_n j_n(k_{L1}r) + E_n y_n(k_{L1}r)]P_n(\cos\theta), \tag{A.6}$$

$$\Psi_1 = \sum_{n=0}^{\infty} i^n(2n+1)[D_n j_n(k_{S1}r) + F_n y_n(k_{S1}r)]P_n(\cos\theta), \tag{A.7}$$

where y_n is the spherical Neuman function of order n. The potentials for waves in the core are

$$\Phi_2 = \sum_{n=0}^{\infty} i^n(2n+1)G_n j_n(k_{L2}r)P_n(\cos\theta), \tag{A.8}$$

$$\Psi_2 = \sum_{n=0}^{\infty} i^n(2n+1)H_n j_n(k_{S2}r)P_n(\cos\theta). \tag{A.9}$$

At two boundaries $r = a$ and $r = b$, the following continuity conditions for displacement and stress components must be satisfied;

$$u_{re} = u_{r1}, \quad u_{\theta e} = u_{\theta 1}, \quad \tau_{rre} = \tau_{rr1}, \quad \tau_{r\theta e} = \tau_{r\theta 1} \quad \text{at } r = b, \tag{A.10}$$

$$u_{r1} = u_{r2}, \quad u_{\theta 1} = u_{\theta 2}, \quad \tau_{rr1} = \tau_{rr2}, \quad \tau_{r\theta 1} = \tau_{r\theta 2} \quad \text{at } r = a, \tag{A.11}$$

Substituting the displacement and stress components derived from their relations to potentials in the spherical coordinate system yields an 8×8 system linear equation for the unknown coefficients and thus the scattering coefficients A_n can be determined by solving the system linear equation. The forward scattering amplitude is

$$f_{\mathrm{L}}(0) = \frac{1}{i\langle k_{\mathrm{L}}\rangle}\sum_{n=0}^{\infty}(2n+1)A_n. \tag{A.12}$$

When wavelengths of all associated waves are much larger than the outer radius of the sphere ($\langle k_L \rangle b, \langle k_S \rangle b \ll 1$) the spherical wave functions can be expanded asymptotically in polynomial series form (Morse and Feshbach, 1953), that is,

$$j_n(z) \approx \frac{n!(2z)^n}{(2n+1)!}\left[1 - \frac{z^2}{2(2n+3)}\right], \tag{A.13}$$

$$h_0(z) \approx -\frac{i}{z}(1 + iz), \tag{A.14}$$

$$h_n(z) \approx -\frac{i(2n)!}{2^n n! z^{n+1}}\left[1 + \frac{z^2}{2(2n-1)}\right] \quad (n \geqslant 1). \tag{A.15}$$

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